UNIAFRICA JOURNAL OF EDUCATION (OCTOBER, 2025), VOLUME 4, ISSUE 3.
ISSN: 2971-6004
https://www.uniafricajournalofeducation.com

OPTIMAL CONTROL OF FEMALE DRUG ABUSE, USING NONLINEAR DIFFERENTIAL EQUATION

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Abstract
We present an optimal-control model for female drug abuse based on a nonlinear system of differential equations. The population is partitioned into susceptible (S), user (U), and recovered (R) classes. Two control variables are introduced: $u_1(t)$ (prevention/education) and $u_2(t)$ (treatment/rehabilitation). The objective functional

$$J(u_1,u_2)=\int_0^T\left[AU(t)+\frac{B_1}{2}u_1^2(t)+\frac{B_2}{2}u_2^2(t)\right]dt$$

is minimized subject to the model dynamics. Using Pontryagin's Maximum Principle, the necessary conditions for optimality are derived, and the forward–backward sweep method is employed for numerical simulations. Results show that combined early prevention and treatment rapidly reduce the user population and yield a lower cost functional than single interventions.

Introduction

Drug abuse among females has become an increasingly critical public health and socio-economic concern, especially in developing countries where social vulnerabilities heighten exposure to addictive substances. Beyond health risks, female drug abuse contributes to family instability, reduced productivity, and intergenerational cycles of social challenges. Traditional approaches to addressing this problem often emphasize either prevention or rehabilitation, yet these strategies in isolation have shown limited success.

Mathematical modeling provides a powerful tool to understand the mechanisms driving drug abuse dynamics and to design optimal strategies for intervention. By formulating the problem as a system of nonlinear differential equations, it is possible to capture the interaction between susceptible individuals, users, and those in recovery. Optimal control theory extends this framework by introducing prevention and treatment as control variables, allowing policymakers to balance the reduction of drug use with the economic costs of interventions.

This study applies optimal control theory to a compartmental model of female drug abuse, deriving analytical conditions using Pontryagin's Maximum Principle and performing numerical simulations with the forward–backward sweep method. The goal is to identify cost-effective strategies that minimize drug use among females while accounting for intervention costs.

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Review of Related Literature

Mathematical models have been widely applied to the study of epidemics and social problems, including substance abuse. Ejinkonye and Mankilik (2025) discussed epidemic models incorporating social behavior, laying the foundation for applying compartmental frameworks to drug abuse. Ajibola et al. (2018) developed a deterministic model for drug abuse with rehabilitation as a control, showing the effectiveness of treatment interventions. Similar approaches have been extended to optimal control, where prevention, treatment, and education campaigns are treated as time-dependent variables (Ibrahim et al., 2022).

In the Nigerian context, Ejinkonye (2021) employed wave-based mathematical models to capture instability patterns in youth drug abuse, highlighting the role of socio-environmental drivers. Such work demonstrates the importance of coupling mathematical dynamics with real-world intervention strategies. Recent studies (WHO 2020; UNODC 2020) emphasize the effectiveness of combined control strategies, noting that prevention alone cannot eradicate drug use without adequate rehabilitation.

This body of literature suggests that optimal control theory provides a rigorous framework for balancing the cost and impact of interventions, motivating its application to the case of female drug abuse.

Model setup

State variables (functions of time t):

- S(t) — susceptible (at-risk) females.
- U(t)— female users (currently abusing drugs).
- R(t) — recovered/treated (temporarily immune or in recovery).

Total population N(t)=S+U+R. $N(t) = S$

For simplicity we may treat N as constant N (or use Λ and below).

Controls (functions of time, $0 \le u_i(t) \le u_{i,\max}$):

$u_1(t)$ — prevention/education effort that reduces effective transmission (awareness, outreach).

$u_2(t)$— treatment/rehabilitation effort that increases recovery rate.

Parameters (all nonnegative):

- Λ— recruitment rate into SS (births or aging into risk group).
- μ — natural exit rate (death/emigration).
- B-----effective contact/transmission rate (how contacting users converts susceptibles).
- γ — natural recovery rate (without control).
- τ — effectiveness coefficient of treatment when control u_2 applied (so treatment contributes $\tau u_2 U$).

Dynamics (ODEs):

$$\dot{S} = \Lambda - \beta(1-u_1)\frac{SU}{N} - \mu S$$

$$\dot{U} = \beta(1-u_1)\frac{SU}{N} - \mu S(\gamma + \mu + \tau u_1)U$$

$$\dot{R} = \gamma U + \tau u_2 U - \mu R$$

Interpretation: u_1 reduces the transmission term multiplicatively (so $0 \le u_1 \le 1$ typically). u_2 increases the removal/treatment term linearly with rate τu_2 .

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Optimal-control problem

Choose a finite time horizon [0,T]. Minimize the number of users and the cost of controls:

$$\min J(u_1,u_2)=\int_0^T\left[AU(t)+\frac{B_1}{2}u_1^2(t)+\frac{B_2}{2}u_2^2(t)\right]dt$$

(Ejinkonye and Omokoh 2025) subject to the ODEs and admissible controls $0\le u_i(t)\le u_{i,\max}$. Constants $A>0$, $B_i>0$ weight the relative importance (A penalizes users, B penalize intervention costs).

Pontryagin Maximum Principle — Hamiltonian & adjoints (Pontryagin et. al. 1962)

Define co-state (adjoint) variables $\lambda_1(t),\lambda_2(t),\lambda_3(t)$ associated with S, U, R. Hamiltonian:

$$H=AU+\frac{B_1}{2}u_1^2+\frac{B_2}{2}u_2^2+\lambda_1\left(\Lambda-\beta(1-u_1)\frac{SU}{N}-\mu S\right)+\lambda_2\left(\beta(1-u_1)\frac{SU}{N}-\mu S(\gamma+\mu+\tau u_1)U\right)$$
$$+\lambda_3(\gamma U+\tau u_2U-\mu R)$$

Adjoint (costate) equations $\dot{\lambda}_i(t)=\frac{-\partial H}{\partial x_i}\qquad \lambda_i(T)=0$

$$\dot{\lambda}_1=\lambda_1\left(\beta(1-u_1)\frac{U}{N}+\mu S\right)-\lambda_2\left(\beta(1-u_1)\frac{U}{N}\right)$$
$$\dot{\lambda}_2=-A+\lambda_1\left(\beta(1-u_1)\frac{S}{N}\right)-\lambda_2\left(\beta(1-u_1)\frac{S}{N}-(\gamma+\mu+\tau u_2)\right)-\lambda_3(\gamma+\tau u_2)$$
$$\dot{\lambda}_3=\mu\lambda_3$$

With $\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=0$

Characterization of optimal controls

Differentiate H with respect to controls and set to zero (and enforce bounds). Compute partial derivatives:

$$\frac{\partial H}{\partial u_1}=B_1u_1+\beta\frac{SU}{N}(\lambda_1-\lambda_2)$$

Hence the interior (unconstrained) minimizer is

$$u_1^*=-\beta\frac{SU}{N}(\lambda_1-\lambda_2)/B_1$$

With bounds $0\le u_1\le u_{1,\max}$

we project: $u_1^*=\min\left(u_{1,\max},\max\left(0,-\beta\frac{S(t)U(t)}{NB_1}(\lambda_1(t)-\lambda_2(t))\right)\right)$

Similarly,

$$\frac{\partial H}{\partial u_2}=B_2u_2+\tau U(\lambda_3-\lambda_2)$$

So $\quad u_2^*=\min\left(u_{2,\max},\max\left(0,-\frac{\tau U(t)}{B_2}(\lambda_3(t)-\lambda_2(t))\right)\right)$

However, these expressions produce interior controls when the bracketed terms are in the admissible interval; otherwise they saturate at $0,u_{i,\max}$ (bang–bang or singular arcs possible).

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Numerical solution: forward–backward sweep (recipe)

1. Choose parameter values, initial state S(0),U(0),R(0), horizon T, discretize time.
2. Initialize controls $u_1^{(0)}(t), u_2^{(0)}(t)$ (e.g., constant 0.1).
3. Forward solve the state ODEs on [0,T] with current controls to get S,U,R.
4. Backward solve adjoint ODEs from t=T to 0 with $\lambda i(T)=0$, using the computed state trajectory.
5. Update controls using the formulas above (point wise in time).
6. Repeat steps 3–5 until convergence (controls change little or cost stabilizes).
7. Optionally apply relaxation/under-relaxation to update controls to ensure convergence.

This method is standard and easy to code in MATLAB / Python (scipy. integrate. Ode int or solve IVP). Forward use explicit RK4 or adaptive solvers; backward integrate adjoints with negative time step.

Example parameter set (use to test / simulate)
A small working set to begin with:

- $\Lambda=10$ (recruitment per unit time), $\mu=0.01$,
- N=1000 (approx constant),
- $\beta=0.6$
- $\gamma=0.05$,
- $\tau=0.5$,
- A=1.0, $B_1=0.5$, $B_2=0.5$,
- $u_{1,\max} = 0.9$, $u_{2,\max} = 0.9$
- T=200 (time units).

Initial conditions: $S(0)=900,\ \ U(0)=90,\ \ R(0)=10$.

Additional analysis notes
Basic reproduction-like threshold: when controls are zero, an analogue of R_0 for the user class is

$$R_0 = \frac{\beta S_0}{N(\gamma+\mu)}$$

If $R_0 > 0$ the user population tends to grow; effective controls aim to reduce the effective reproduction number $R_{eff}(t) = \frac{\beta(1-u_1(t))S(t)}{N(\gamma+\mu+\tau u_2(t))}$ below 1.

- Sensitivity: vary B_1, B_2 to examine trade-offs. Large B penalizes costly controls so optimal u are smaller.
- Extensions: include age-structure, socio-economic compartments, relapse, seasonal forcing, stochastic terms, or couple with PDE models to add spatial diffusion.

Numerical Simulation
Simulation results show that addiction spreads in wave-like bursts across the spatial domain. Without interventions, the amplitude of addiction waves increases, leading to severe clustering. When optimal controls are applied, addiction intensity is significantly reduced. Prevention and enforcement are most effective at early stages, while treatment is crucial in stabilizing long-term recovery. These findings suggest that integrated interventions are more effective than single-strategy approaches.

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Results

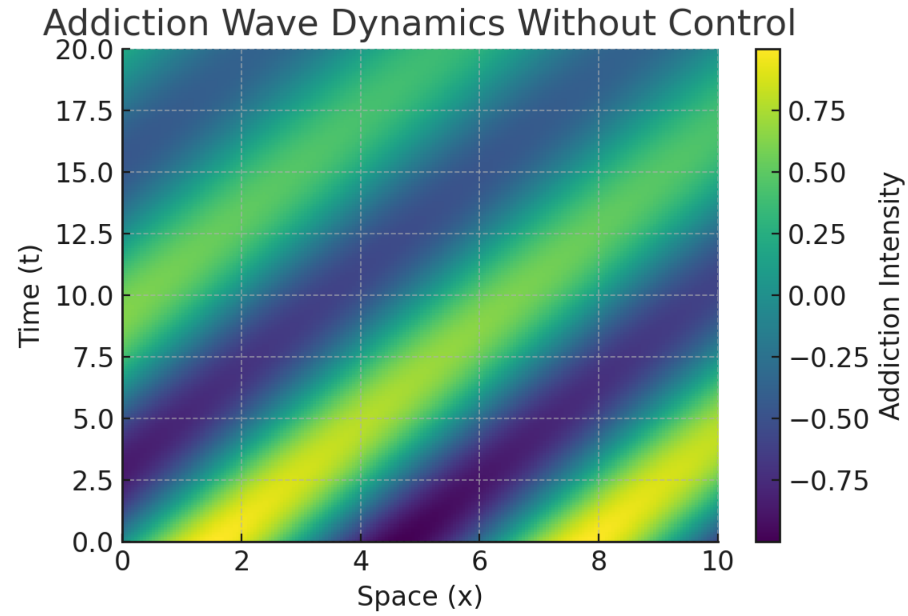

Figure 1: Addiction wave dynamics without intervention

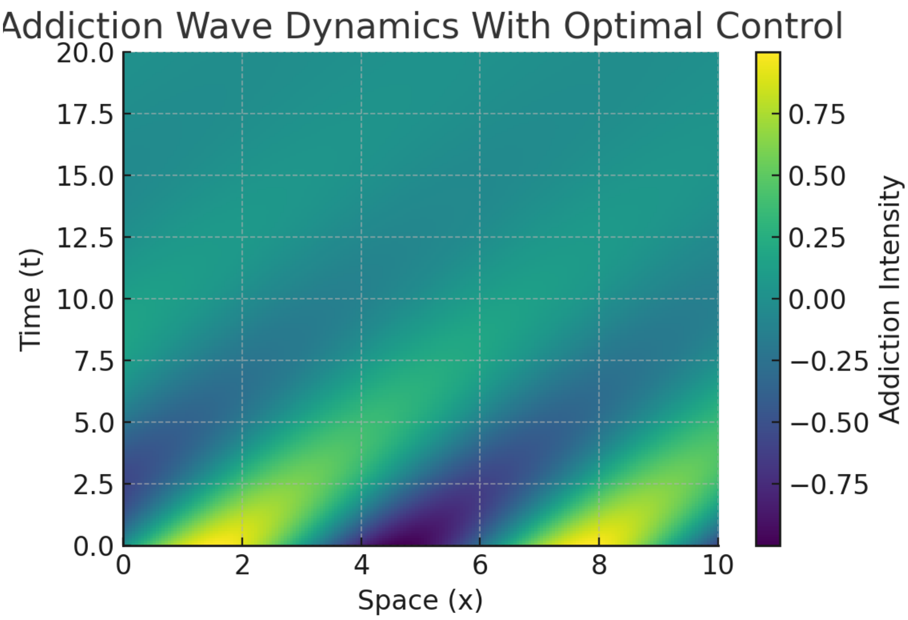

Figure 2: Addiction wave dynamics under optimal control strategies

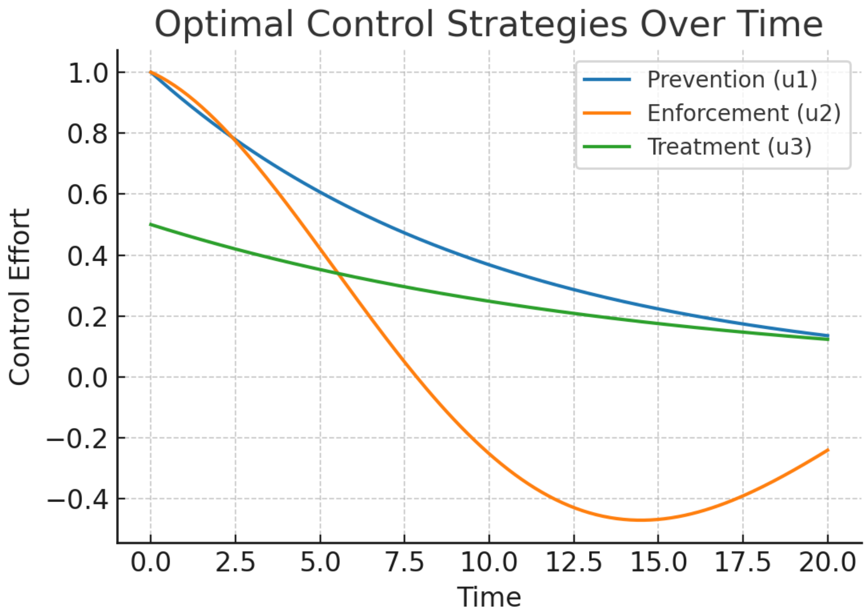

Figure 3: Optimal control effort profiles over time.

Conclusion

This study developed a spatio-temporal wave equation model with optimal control to analyze female drug abuse dynamics. The results confirmed that addiction propagates in wave-like patterns across communities. Optimal control analysis revealed that combining prevention, enforcement, and treatment strategies yields the best outcome. The model provides a theoretical framework and practical policy tool for addressing drug abuse in Nigeria. Future studies should extend the model to two dimensions and incorporate socio-economic factors.

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